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A Coloring Algorithm for Triangle-Free Graphs

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Abstract. We give a randomized algorithm that properly colors the vertices of a triangle-free graph G on n vertices using $O(\Delta(G)/\log\Delta(G))$ colors, where $\Delta(G)$ is the maximum degree of G. The algorithm takes $O(n\Delta^2(G)\log\Delta(G))$ time and succeeds with high probability, provided $\Delta(G)$ is greater than $log^{1+\epsilon}n$ for a positive constant ϵ. The number of colors is best possible up to a constant factor for triangle-free graphs. As a result this gives an algorithmic proof for a sharp upper bound of the chromatic number of a triangle-free graph, the existence of which was previously established by Kim and Johansson respectively.

1 Introduction

A *proper vertex coloring* of a graph is an assignment of colors to all vertices such that adjacent vertices have distinct colors. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors required for a proper vertex coloring. Finding the chromatic number of a graph is NP-Hard [10]. Approximating it to within a polynomial ratio is also hard [15]. For general graphs, $\Delta(G)+1$ is a trivial upper bound. Brooks' Theorem [7] shows that $\chi(G)$ can be $\Delta(G)+1$ only if G has a component which is either a complete subgraph or an odd cycle.

A natural question is: can this bound be improved for graphs without large complete subgraphs? In 1968, Vizing [22] had asked what the best possible upper bound for the chromatic number of a triangle-free graph was. Borodin and Kostochka [6], Catalin [8], and Lawrence [19] independently made progress in this direction; they showed that for a K_4-free graph, $\chi(G) \leq 3(\Delta(G)+2)/4$. On the other hand, Kostochka and Masurova [18], and Bollobás [5] separately showed that there are graphs of arbitrarily large *girth*(length of a shortest cycle) with $\chi(G)$ of order $\Delta(G)/\log\Delta(G)$.

Further progress was made using the *semi-random method* to show that the chromatic number of graphs with large girth is $O(\Delta(G)/\log\Delta(G))$. This technique, also known as the *pseudo-random method*, or the *Rödl nibble*, appeared first in Ajtai, Komlós and Szemerédi [2] and was applied to problems in hypergraph packings, Ramsey theory, colorings, and list colorings [9,13,14,17,21]. In general, given a set S_1, the goal is to show that there is an object in S_1 with a desired property $\mathcal{P}$. This is done by locating a sequence of non-empty subsets $S_1 \supseteq \cdots \supseteq S_\tau$ with S_τ having property $\mathcal{P}$. A randomized algorithm is applied to S_t, which guarantees that S_{t+1} will be obtained with some non-zero(often small) probability. For upper bounds on chromatic number, the semi-random method is used to prove the existence of a proper coloring with a limited number of colors. It does not give an efficient probabilistic algorithm.

In 1995, Kim [16] proved that

$$\chi(G) \leq (1+o(1))\frac{\Delta(G)}{\log\Delta(G)}$$

when G has girth greater than 4. Later on, Johansson [12] showed that

$$\chi(G) \leq O(\frac{\Delta(G)}{\log\Delta(G)})$$

when G is a triangle-free graph(girth greater than 3). Both Kim and Johansson used the semi-random method. Alon, Krivelevich and Sudakov [3], and Vu [23] extended the method of Johansson to prove bounds

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on the chromatic number for graphs in which no subgraph on the set of all neighbors of a vertex has too many edges.

Grable and Panconesi [11] gave an algorithm for properly coloring a Δ-regular graph with girth greater than 4 and $\Delta \geq \log^{1+\epsilon} n$ for a positive constant ϵ. The procedure uses $O(\Delta/\log \Delta)$ colors and has polynomial running time. This is a constructive version of the existential proof of Kim with extra conditions on the degree of the input graph. Section 5 of [11] asserts that the algorithm can be extended to triangle-free graphs. In Section 2.1 we will see a counter-example that shows that their analysis does not work on all triangle-free graphs. In particular, as will be seen, the analysis fails on a complete bipartite graph.

In this paper we give a method for coloring the larger class of triangle-free graphs. Our randomized algorithm properly colors a triangle-free graph G on n vertices with $\Delta(G) \geq \log^{1+\epsilon} n$ for a positive constant ϵ, using $O(\Delta(G)/\log \Delta(G))$ colors in $O(n\Delta^2(G) \log \Delta(G))$ time. The probability of failure goes to 0 as n becomes large.

To analyze our iterative algorithm we identify a collection of random variables. The expected changes to these random variables after a round of the algorithm are written in terms of the values of the random variables before the round. We thus obtain a set of recurrence relations and prove that our random variables are concentrated around the solutions to the recurrence relations.

Our method of analysis resembles that of Kim, and Grable et al. However our algorithm and collection of random variables are different. Also related to our analysis technique is Achlioptas and Molloy's study of the performance of a coloring algorithm on random graphs [1]. They setup recurrence relations as we do, and then use the so-called differential equation method for random graph processes [24].

We describe our algorithm in Section 2. Section 2.1 contains motivation, which is followed by a formal description of the algorithm in Section 2.2. We give an outline of the analysis in Section 3. Section 4 contains some useful lemmas which we are used in Section 5 to give details of the analysis.

2 An Iterative Algorithm for Coloring a Graph

Our algorithm takes as input a triangle-free graph G on n vertices, its maximum degree Δ, and the number of colors to use Δ/k where k is a positive number. It goes through rounds and assigns colors to more vertices each round. Initially all vertices are *uncolored*(no color assigned), at the end we have a proper vertex coloring of G with some probability.

Definition 1. *Let t be a natural number. We define the following:*

G_t *The graph induced on G by the vertices that are uncolored at the beginning of round t.*

$N_t(u)$ *The set of vertices adjacent to vertex u in G_t. That is, the set of uncolored neighbors of u at the beginning of round t.*

$S_t(u)$ *The list of colors that may be assigned to vertex u in round t, also called the* palette *of u. For all u in $V(G)$,*

$$S_0(u) = \{1, \ldots, \Delta/k\}.$$

$D_t(u,c)$ *The set of vertices adjacent to u that may be assigned color c in round t. That is,*

$$D_t(u,c) := \{v \in N_t(u) | c \in S_t(v)\}.$$

It will be useful to define variables for the sizes of the sets $N_t(u)$, $S_t(u)$, and $D_t(u,c)$.

Definition 2.

$$\begin{aligned} \eta_t(u) &= |N_t(u)| \\ s_t(u) &= |S_t(u)| \\ d_t(u,c) &= |D_t(u,c)| \end{aligned}$$

Observe that for every round t, vertex u in $V(G)$, and color c in $\{1,\ldots,\Delta/k\}$,

$$d_t(u,c), \eta_t(u) \leq \Delta, \;\; s_0(u) = \Delta/k, \;\; s_t(u) \leq \Delta/k.$$

2.1 A Sketch of the Algorithm and the Ideas behind its Analysis

We start by considering an algorithm that works for Δ-regular graphs with girth greater than 4. The rounds of this algorithm will go through two stages. The number of colors in the palette of each uncolored vertex will be greater than the number of uncolored adjacent vertices at the end of the second stage with some probability. Then the partial coloring of the graph can be completed easily to give a proper coloring.

Let (η_t), (d_t), and (s_t) be sequences defined recursively as

$$\begin{array}{ll} \eta_0 := \Delta & \eta_{t+1} := \eta_t(1 - c_1\frac{s_t}{d_t}) \\ d_0 := \Delta & d_{t+1} := d_t(1 - c_1\frac{s_t}{d_t})c_2 \\ s_0 := \Delta/k & s_{t+1} := s_t c_2 \end{array}$$

where c_1 and c_2 are constants between 0 and 1, which are determined by the analysis of the algorithm.

First Stage

Repeat at every round t, until $d_t/s_t < 1$
Phase I - Coloring Attempt
For each vertex u in G_t:
  Awake vertex u with probabilitity s_t/d_t.
  If *awake*, assign to u a color chosen from $S_t(u)$ uniformly at random.

Phase II - Conflict Resolution
For each vertex u in G_t:
  If u *is awake*,
    uncolor u if an adjacent vertex is assigned the same color in the coloring attempt phase.
  Remove from $S_t(u)$, all colors assigned to adjacent vertices.
  $S_{t+1}(u) = S_t(u)$.
end repeat

Observe that $d_0/s_0 = k$ and

$$\frac{d_{t+1}}{s_{t+1}} = \frac{d_t}{s_t} - c_1.$$

In $O(k)$ rounds d_t/s_t will be less that 1, and this marks the end of the first stage.

Second Stage We change the recursive equations for η_t, d_t, and s_t.

$$\begin{aligned} \eta_{t+1} &:= \eta_t(1 - c_1 e^{c_2 d_t/s_t}) \\ d_{t+1} &:= d_t(1 - c_1 e^{c_2 d_t/s_t})e^{-c_2 d_t/s_t} \\ s_{t+1} &:= s_t e^{-c_2 d_t/s_t} \end{aligned}$$

All uncolored vertices are woken up at every round. In this stage, η_t decreases much faster than s_t and the repeat-until loop is repeated until

$$\eta_t < s_t.$$

Otherwise the second stage is the same as the first.

The two stage algorithm is derived from Grable et al. Their analysis tells us that if graph G has girth greater than 4 and $\Delta \geq \log^{1+\epsilon} n$ for some positive constant ϵ, then there are constants c_1 and c_2 less than 1 such that at each round t, $\forall u \in V(G_t), \forall c \in S_t(u)$

$$\eta_t(u) = \eta_t(1 + o(1)), \quad s_t(u) = s_t(1 + o(1)), \quad d_t(u, c) = d_t(1 + o(1)) \tag{1}$$

with probability $1 - o(1)$. The equations above imply that at the end of the second stage $s_t(u) > \eta_t(u)$ for all uncolored vertices u with probability approaching 1 as n approaching ∞.

The change we have made to Grable et al. so far, is that we remove all colors temporarily assigned to neighbors of a vertex from its palette, instead of removing only those assigned permanently. This simple but powerful idea, adapted from Kim [16], will waste a few colors from the palettes, and simplify our algorithm and its analysis significantly.

A counter-example to demonstrate that Grable et al. does not work on graphs with girth greater than 3. The analysis for the above algorithm is probabilistic and proves the property in equation (1) by induction, showing concentration of the variables around their expectations. It fails for graphs with 4-cycles. An example illustrates why: Consider a vertex u whose 2-neighborhood, the graph induced by vertices within distance 2 of u, is the complete bipartite graph $K_{\Delta,\Delta}$ with partitions X and Y. Suppose that u and another vertex v are in X. If v is colored with c in round 0 while u remains uncolored, then the set $D_1(u, c) = \emptyset$; this violates equation (1) since $d_1 \geq 1$ if for example $k \geq 2$ and $\Delta \geq 2/c_2$. So, when the graph has 4-cycles, $d_{t+1}(u, c)$ is not necessarily concentrated around its expectation given the state of the algorithm at the beginning of round t. Maintaining equation (1) is crucial for the proof in Grable et al., and this violation is the error we mentioned in the introduction; their algorithm and analysis do not work for triangle-free graphs in general!

We must modify the algorithm in two more ways.

First Modification: A technique for coloring graphs with 4-cycles. While $d_t(u, c)$ is not concentrated enough when the graph has 4-cycles, our analysis will show that the average of $d_{t+1}(u, c)$ over all colors in the palette of a vertex u is concentrated enough. How does this benefit us? Markov's famous inequality may be interpreted as: a list of s positive number which average d has at most s/q numbers larger than qd for any positive number q. We modify the algorithm so that at the end of each round t, every vertex u removes from its palette every color c with $d_{t+1}(u, c)$ larger than qd_{t+1} for some constant q larger than 1. Look at what happens in round $t = 1$. By a straightforward application of Markov's inequality, instead of equation (1) we will have the less stringent property: $\forall u \in V(G_t), \forall c \in S_t(u)$

$$\eta_t(u) \leq \eta_t(1 + o(1)), \quad s_t(u) \geq \frac{q-1}{q} s_t(1 - o(1)), \quad d_t(u, c) \leq qd_t(1 + o(1)). \tag{2}$$

with probability $1 - o(1)$. In fact, using a generalization of Markov's inequality, the analysis will show that with a few more modifications our algorithm maintains a slightly stronger property(still weaker than equation (1)).

Equation (1) implies that the $\eta_t(u)$, $s_t(u)$, and $d_t(u)$ at all uncolored vertices u are about the same. It is a strong statement and helps in the proofs, but is too much to maintain on graphs with 4-cycles. Equation (2) is weaker and is guaranteed by our algorithm and what is more, it is sufficient to guarantee that all uncolored vertices at the end of the second stage have palettes with more colors than the number of uncolored adjacent vertices. This is a key idea in our algorithm.

Second Modification: Using independent random variables for easier analysis. Instead of waking up a vertex with some probability, and then choosing a color from its palette uniformly at random; for each uncolored vertex u and color c in its palette, we will assign c to u independently with some probability. In case multiple colors remain assigned to the vertex after the conflict resolution phase, we will arbitrarily choose one of them to permanently color the vertex. This modification, adapted from Johansson [12], will make concentration of our random variables simpler.

Next we provide a formal description of the algorithm we have just motivated.

2.2 A Formal Description of the Algorithm

In each round of the algorithm, some vertices are colored. The details of the coloring procedure vary depending on which of three stages the algorithm is in.

First Stage Let q be a constant greater than 1, and let (η_t), (d_t), and (s_t) be sequences defined recursively as

$$\begin{aligned} \eta_0 &:= \Delta & \eta_{t+1} &:= \eta_t\left(1 - \frac{q-1}{2q^3}e^{-1/q}\frac{s_t}{d_t}\right) \\ d_0 &:= \Delta & d_{t+1} &:= d_t\left(1 - \frac{q-1}{2q^3}e^{-1/q}\frac{s_t}{d_t}\right)e^{-1/q} \\ s_0 &:= \Delta/k & s_{t+1} &:= s_t e^{-1/q}. \end{aligned} \tag{3}$$

For round t, vertex u, and color c,

$$\mathcal{F}_t(u,c) := \{c \text{ is not assigned to any vertex adjacent to } u \text{ in round } t\} \tag{4}$$

is an event in the probability space generated by the random choices of the algorithm in round t, given the state of all data structures at the beginning of the round.

Let

$$Desired\_\mathcal{F}_t := e^{-1/q}.$$

Repeat at every round t, until $d_t/s_t < 1/q^2$
Phase I - Coloring Attempt
For each vertex u in G_t, and color c in $S_t(u)$:
Assign c to u with probability $\frac{1}{q^2}\frac{1}{d_t}$.

Phase II - Conflict Resolution
For each vertex u in G_t:
Phase II.1
Remove from $S_t(u)$, all colors assigned to adjacent vertices.
Phase II.2
For each color c in $S_t(u)$, remove c from $S_t(u)$ with probability

$$1 - min(1, \frac{Desired\_\mathcal{F}_t(u,c)}{Pr(\mathcal{F}_t(u,c)}).$$

If $S_t(u)$ has at least one color which is assigned to u,
then arbirarily pick an assigned color from $S_t(u)$ to permanently color u.

Phase III - Cleanup(discard all colors c with $d_{t+1}(u,c) \gtrsim qd_{t+1}$ from palette)
For each vertex u in G_t:
$S_{t+1}(u) = S_t(u)$.
Let $\alpha = 1 - |S_{t+1}(u)|/s_{t+1}$.
If $\alpha < 0$, then $\alpha = 0$, otherwise if $\alpha > 1/q$, then $\alpha = 1/q$.
Let γ be the smallest number in $[1, \infty)$ so that

$$Average_{c \in S_{t+1}(u)} d_{t+1}(u,c) \leq \frac{1 - q\alpha}{1 - \alpha} \gamma d_{t+1}.$$

Remove all colors c with $d_{t+1}(u,c) \geq q\gamma d_{t+1}$ from $S_{t+1}(u)$.
end repeat

Second Stage We change the recursive equations defining the constants η_t, d_t, and s_t.

$$\begin{aligned} \eta_{t+1} &:= \eta_t(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}) \\ d_{t+1} &:= d_t(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}) e^{-\frac{1}{q}\frac{d_t}{s_t}} \\ s_{t+1} &:= s_t e^{-\frac{1}{q}\frac{d_t}{s_t}} \end{aligned} \tag{5}$$

Also

$$Desired\_\mathcal{F}_t := e^{-\frac{1}{q}\frac{d_t}{s_t}}.$$

All other details of the repeat-until loop of the first stage are the same for the second stage, except that an uncolored vertex u is assigned a color c from its palette with probability

$$\frac{1}{q^2}\frac{1}{s_t},$$

and we repeat until

$$\eta_t < \frac{q-1}{q} s_t.$$

Third Stage(Greedy Coloring) Color each uncolored vertex u, with an arbitrary color from its palette which has not been used to color an adjacent vertex.

3 The Main Theorem

We say that a sequence $x(n)$ is $O(f(n))$ if there is a positive number M such that $|x(n)| \leq M|f(n)|$. All sequences in the big-oh are indexed by n, the number of vertices in graph G. Remember that each occurrence of the big-oh comes with a distinct constant M which may depend on the constant ϵ.

Theorem 1 (Main Theorem). *Given a triangle-free graph G on n vertices with maximum degree $\Delta \geq \log^{1+\epsilon} n$ for a positive constant ϵ, and $q = 7$, there is a positive constant c_ϵ such that starting with $c_\epsilon \Delta / \log \Delta$ colors, our algorithm finds a proper coloring of the graph in $O(n\Delta^2 \log \Delta)$ time with probability $1 - O(1/n)$.*

We need some lemmas to prove the Main Theorem and before that we need the following definition.

Definition 3. $d_t(v) = Average_{c \in S_t(v)} d_t(v, c)$

Lemma 1 (Main Lemma). *Given a triangle-free graph G on n vertices with maximum degree $\Delta \geq \log^{1+\epsilon} n$ for a positive constant ϵ, $q \geq 2$, and $\psi > 1$, there are positive constants α_1 and α_2 such that for the sequences (e_t) defined by*

$$e_0 = 0, \quad e_{t+1} = \alpha_1 (e_t + \sqrt{\frac{\psi}{d_t}} + \sqrt{\frac{\psi}{s_t}}) \quad \text{for } t > 0, \tag{6}$$

and (f_t) defined by

$$f_t = 1 - \alpha_2 t n^2 e^{-\psi}, \tag{7}$$

if $s_t \gg \psi$ and $e_t \ll 1$ at round t, then

$$\forall u \in V(G_t), \exists \alpha \in [0, 1/q], \forall c \in S_t(u),$$

$$\begin{aligned} s_t(u) &\geq (1 - \alpha) s_t (1 - e_t) \\ d_t(u) &\leq \frac{1 - q\alpha}{1 - \alpha} d_t (1 + e_t) \\ d_t(u, c) &\leq q d_t (1 + e_t) \\ \eta_t(u) &\leq \eta_t (1 + e_t) \end{aligned}$$

with probability greater than f_t.

We will prove the Main Lemma in Section 5 and assume it in this section. Using it we can immediately conclude the following.

Corollary 1. *Given the setup of the Main Lemma(Lemma 1), if $s_t \gg \psi$ and $e_t \ll 1$ at round t, then $\forall u \in V(G_t)$,*

$$\begin{aligned} s_t(u) &\geq \frac{q-1}{q} s_t (1 - e_t) \\ d_t(u) &\leq d_t (1 + e_t) \end{aligned}$$

with probability greater than f_t.

Lemma 2. *The first stage finishes in $O(k)$ rounds.*

Proof. By the definition of sequences (d_t) and (s_t) in equation (3), we have

$$\begin{aligned} \frac{d_{t+1}}{s_{t+1}} &= \frac{d_t}{s_t} (1 - \frac{q-1}{2q^3} \frac{s_t}{d_t} e^{-1/q}) \\ &= \frac{d_t}{s_t} - \frac{q-1}{2q^3} e^{-1/q}. \end{aligned}$$

Since $\frac{d_0}{s_0} = k$ we get $\frac{d_{t_1}}{s_{t_1}} \leq \frac{1}{q^2}$ for some round $t_1 = O(k)$. $\square$

Let

$$t_1 = O(k)$$

be the last round of the first stage. Then the following lemma is a straightforward application of equation (3).

Lemma 3.

$$s_{t_1} = \frac{\Delta}{k} e^{-O(k)} \textit{ and } \exists c > 0 \textit{ such that if } k \le c \log \Delta, \textit{ then } s_{t_1} \gg 1.$$

3.1 The Second Stage: Controlling the ratio of available colors to uncolored neighbors.

We must show that η_t decreases significantly faster than s_t in the second stage. Let

$$\rho = 1 - \frac{q-2}{2q^3}.$$

Lemma 4.

$$\eta_{t_1+t} < \eta_{t_1} \rho^t$$

Proof. By the definition of sequence (η_t) in equation (5), we have

$$\begin{aligned}
\eta_{t+1} &= \eta_t \left(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}\right) \\
&< \eta_t \left(1 - \frac{q-1}{2q^3}\left(1 - \frac{1}{q}\frac{d_t}{s_t}\right)\right) && \langle e^x \ge 1 + x \rangle \\
&< \eta_t \left(1 - \frac{q-1}{2q^3}\left(1 - \frac{1}{q}\right)\right) && \langle \frac{d_t}{s_t} < 1 \rangle \\
&< \eta_t \left(1 - \frac{q-1}{2q^3} + \frac{1}{2q^3}\right) \\
&= \eta_t \left(1 - \frac{q-2}{2q^3}\right) \\
&= \eta_t \rho.
\end{aligned}$$

Thus $\eta_{t_1+t} < \eta_{t_1} \rho^t$. □

We now study the ratio d_t/s_t, which appears in the recursive equation (5) for the sequences (η_t), (d_t), and (s_t).

Lemma 5.

$$\frac{d_{t_1+t}}{s_{t_1+t}} < \frac{1}{q^2} \rho^t$$

Proof. By the definition of sequences (d_t) and (s_t) in equation (5), we have

$$\frac{d_{t+1}}{s_{t+1}} = \frac{d_t}{s_t}\left(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}}\right).$$

Since the ratio of d_{t+1}/s_{t+1} to d_t/s_t is the same as the ratio of η_{t+1} to η_t, we use Lemma 4 to conclude that

$$\frac{d_{t_1+t}}{s_{t_1+t}} < \frac{d_{t_1}}{s_{t_1}} \rho^t < \frac{1}{q^2} \rho^t.$$

□

Lemma 6. *If q is greater than 2, then*

$$s_{t_1+t} \geq s_{t_1}(1 - \frac{4}{q-2}).$$

Proof. By the definition of sequence (s_t) in equation (5), we have

$$\begin{aligned} s_{t+1} &= s_t e^{-\frac{1}{q}\frac{d_t}{s_t}} \\ &\geq s_t e^{-\frac{1}{q}\frac{d_t}{s_t}} \\ &\geq s_t(1 - \frac{1}{q}\frac{d_t}{s_t}). \end{aligned}$$

Thus

$$\begin{aligned} s_{t_1+t} &\geq s_{t_1}\prod_{i=0}^{t}(1 - \frac{1}{q}\frac{d_t}{s_t}) \\ &\geq s_{t_1}\prod_{i=0}^{t}(1 - \frac{1}{q^3}\rho^i) && \langle \text{Lemma 5} \rangle \\ &\geq s_{t_1} exp(-\frac{2}{q^3}\sum_{i=0}^{t}\rho^i) && \langle \text{since } q > 2 \rangle \\ &\geq s_{t_1} exp(-\frac{2}{q^3}\frac{1}{1-\rho}) && \langle \text{sum of geometric series} \rangle \\ &= s_{t_1} exp(-\frac{4q^3}{q^3(q-2)}) \\ &= s_{t_1} exp(-\frac{4}{q-2}) \\ &\geq s_{t_1}(1 - \frac{4}{q-2}). \end{aligned}$$

□

Lemma 7. *For any $\delta > 0$, there is a $c_\delta > 0$ such that if $q > 6$, $k \leq c_\delta \log \Delta$, and $\Delta \gg 1$, then the second stage takes at most $\delta \log \Delta$ rounds.*

Proof. By Lemma 3, we have

$$s_{t_1} = \frac{\Delta}{k}e^{-O(k)}.$$

Since $q > 6$, the above equation and Lemma 6 imply that

$$s_{t_1+t} = \Omega(\frac{\Delta}{k}e^{-O(k)})$$

for all t. Also, by Lemma 4, we have

$$\eta_{t_1+t} < \Delta\rho^t.$$

Since $\Delta \gg 1$, given a δ it is straightforward to find a $c_\delta > 0$ so that if $k \leq c_\delta \log \Delta$, then after $\delta \log \Delta$ rounds $\eta_t < \frac{q-1}{q}s_t$. □

3.2 Bounding the Error Estimate in all Concentration Inequalities

Now we look at d_t, which is used to bound the error term e_t. Let

$$\mu = 1 - \frac{1}{2q^2}.$$

Lemma 8.

$$d_{t_1+t} \geq d_{t_1}(1 - \frac{4}{q-2})\mu^t$$

Proof. By the definition of sequence (d_t) in equation (5), we have

$$\begin{aligned} d_{t+1} &= d_t(1 - \frac{q-1}{2q^3}e^{-\frac{1}{q}\frac{d_t}{s_t}})e^{-\frac{1}{q}\frac{d_t}{s_t}} \\ &\geq d_t(1 - \frac{1}{2q^2})e^{-\frac{1}{q}\frac{d_t}{s_t}} \\ &= d_t\mu\, e^{-\frac{1}{q}\frac{d_t}{s_t}}. \end{aligned}$$

Combining the inequality above with the definition of sequence (s_t) in equation (5), we get

$$\frac{d_{t+1}}{d_t} \geq \mu\frac{s_{t+1}}{s_t}.$$

We now use Lemma 6 to conclude that

$$\frac{d_{t_1+t}}{d_{t_1}} \geq \mu^t\frac{s_{t_1+t}}{s_{t_1}} \geq \mu^t(1 - \frac{4}{q-2}).$$

□

Let t_2 be the number of rounds spent in the second stage.

Lemma 9. *There is a positive constant α such that in the first two stages of our algorithm,*

$$e_t \leq \alpha^t O(\sqrt{\frac{k\ e^{O(k)}\psi}{\Delta\mu^{t_2}}}).$$

Proof. By Lemma 3, we have

$$s_{t_1} = \frac{\Delta}{k}e^{-O(k)}.$$

Note that in equation (6), the recurrence for e_t, the largest term is $O(\sqrt{\psi/s_t})$ in the first stage, while $O(\sqrt{\psi/d_t})$ is larger in the second. At round t_1, the algorithm moves to the second stage and $d_{t_1} = \Theta(s_{t_1})$. The greedy stage starts at round $t_1 + t_2$. Since both sequences (d_t) and (s_t) are decreasing, we use Lemma 8 to conclude that

$$O(\sqrt{\frac{\psi}{d_{t_1+t_2}}}) = O(\sqrt{\frac{\psi}{s_{t_1}\mu^{t_2}}}) = O(\sqrt{\frac{k\ e^{O(k)}\psi}{\Delta\mu^{t_2}}})$$

is the maximum this error term can be. Thus we can simplify the recurrence for e_t to

$$e_{t+1} = O(e_t + \sqrt{\frac{k\ e^{O(k)}\psi}{\Delta\mu^{t_2}}}).$$

Since $e_0 = 0$, a simple upper bound for e_t is given by

$$e_t \leq \alpha^t O(\sqrt{\frac{k\ e^{O(k)}\psi}{\Delta\mu^{t_2}}})$$

where α is some positive constant. □

Lemma 10. *Given a triangle-free graph G on n vertices with maximum degree $\Delta \geq \log^{1+\epsilon} n$ for a positive constant ϵ, and $q = 7$, there is a positive constant c_ϵ such that, with Δ/k colors where $k \leq c_\epsilon \log \Delta$, our algorithm reaches the greedy stage at round $\tau = O(\log \Delta)$ with $e_\tau \ll 1$, and $\forall u \in V(G_\tau)$*

$$s_\tau(u) \geq \frac{q-1}{q} s_\tau(1-e_\tau)$$
$$\eta_\tau(u) \leq \eta_\tau(1+e_\tau)$$

with probability $1 - O(1/n)$.

Proof. Let $\psi = 3\log n$, and let $\tau = t_1 + t_2$ be the number of rounds to reach the greedy stage. Using Lemma 9, we get

$$\exists \alpha, \beta > 0, \quad e_\tau \leq \alpha^{t_1} \beta^{t_2} O(\sqrt{\frac{ke^{O(k)}\psi}{\Delta}}).$$

Since $\Delta \geq \log^{1+\epsilon} n$, it is straightforward to show that

$$\exists c_1 > 0, \quad e_\tau \ll 1$$

if $t_1, t_2, k \leq c_1 \log \Delta$. Since $t_1 = O(k)$, by Lemma 7

$$\exists c_2 > 0, \quad t_1, t_2 \leq c_1 \log \Delta, \quad \text{and} \quad s_\tau \gg \psi$$

if $k \leq c_2 \log \Delta$.

Let $c_\epsilon = min(c_1, c_2)$. The above computations show that if $k \leq c_\epsilon \log \Delta$, then $\tau = t_1 + t_2 = O(k + \log \Delta) = O(\log \Delta)$ and $e_\tau \ll 1$. Applying Corollary 1 completes the proof. □

We may now prove the Main Theorem.

Proof (of the Main Theorem). Using Lemma 10, we only need to compute the time complexity of our algorithm. The number of steps taken in the greedy coloring stage is dominated by that of the first and second stages. In each round of these stages, we compute the probability of the event $\mathcal{F}_t(u, c)$ defined in equation (4). This requires iterating through each uncolored vertex u and color c in its palette, and examining all adjacent vertices.

Thus each round takes $O(n\Delta^2)$ steps and by Lemma 10, the algorithm requires $\tau = O(\log \Delta)$ rounds to reach the greedy coloring stage. With probability $1 - O(1/n)$, we get

$$\forall u \in V(G_\tau), \quad \eta_t(u) < s_\tau(u),$$

which implies that the greedy coloring stage will successfully complete the coloring. □

4 Several Useful Inequalities

Now we look at some preliminaries which will be used in the proof details. The next lemma describes what happens to the average value of a finite subset of real numbers when large elements are removed. As shown in the statement of the lemma, it implies Markov's Inequality [4].

Lemma 11. *Consider a set of positive real numbers of size n and average value μ. If we remove αn elements with value atleast $q\mu$ for some $q > 1$, then the remaining points have average*

$$\mu' \leq \mu \frac{1 - q\alpha}{1 - \alpha}.$$

In particular, $\alpha \leq \frac{1}{q}$ since $\mu' \geq 0$.

Proof. The conclusion is obtained by a trivial manipulation of the following inequality which relates μ and μ'.

$$q\mu\alpha + \mu'(1-\alpha) \leq \mu$$

□

The next lemma describes what happens when we add large elements to a finite subset of real numbers.

Lemma 12. *Given the setup of Lemma 11, if we add αn points with value $q\mu$ to the sample, then the resulting larger sample has average*

$$\mu' = \mu \frac{1+q\alpha}{1+\alpha}$$

Proof. The conclusion is easily obtained from the following equation relating μ and μ'.

$$\mu'(1+\alpha) = \mu + q\mu\alpha$$

□

We use the following lemma for computations with error factors.

Lemma 13. *Let (A_n) be a sequence such that $0 < A_n < c < 1$(where c is a constant), and let (e_n) be another sequence. Then*

$$1 - A_n(1+e_n) = (1-A_n)(1+O(e_n))$$

Proof.

$$\begin{aligned}
1 - A_n(1+e_n) &= (1-A_n)(1+e_n) - e_n \\
&= (1-A_n)(1+e_n) - (1-A_n)\frac{e_n}{(1-A_n)} \\
&= (1-A_n)(1+e_n) - (1-A_n)O(e_n) \\
&= (1-A_n)(1+O(e_n))
\end{aligned}$$

□

We use the following version of Azuma's inequality [20] to prove concentration of random variables.

Theorem A (Azuma's inequality) *Let X be a random variable determined by n trials $T_1, \ldots, T_n$, such that for each i, and any two possible sequences of outcomes $t_1, \ldots, t_i$ and $t_1, \ldots, t_{i-1}, t'_i$:*

$$|E[X|T_1 = t_1, \ldots, T_i = t_i] - E[X|T_1 = t_1, \ldots, T_i = t'_i]| \leq \alpha_i$$

then

$$Pr(|X - E[X]| > t) \leq 2e^{-t^2/(\sum \alpha_i^2)}$$

5 Proof of the Main Lemma

Proof (of the Main Lemma). The proof is by induction on the round number t using the lemmas that follow. The base case, when $t = 0$, is trivially true. Lemmas 18, 19, and 20 give us the induction step for the first stage; Lemmas 25, 26, and 27 give the induction step for the second. □

All events below are in the probability space generated by our randomized algorithm in round $t+1$, given the state of all data structures at the beginning of the round. The following assumptions are the induction hypothesis and are repeatedly used in all the lemmas of this section.

Assumption 1 *Assume*

$$q \geq 2, s_t \gg \psi, e_t \ll 1$$

and

$$\forall u \in V(G_t), \forall c \in S_t(u), \exists \alpha \in [0, 1/q],$$

$$\begin{aligned} s_t(u) &\geq (1-\alpha)s_t(1-e_t) \\ d_t(u) &\leq \frac{1-q\alpha}{1-\alpha} d_t(1+e_t) \\ d_t(u,c) &\leq q d_t(1+e_t) \\ \eta_t(u) &\leq \eta_t(1+e_t). \end{aligned}$$

5.1 Expected Values and Concentration Inequalities for the First Stage

We next consider the state of the palettes just before the cleanup phase of round t.

Definition 4. *Let $\tilde{S}_t(u)$ be the list of colors in the palette of vertex u in round t just before the cleanup phase, and let $\tilde{s}_t(u)$ be the size of $\tilde{S}_t(u)$. That is, $\tilde{S}_t(u)$ is obtained from $S_t(u)$ by removing colors discarded in the conflict resolution phase.*

Consider the event

$$\tilde{\mathcal{S}} = \{\forall u \in V(G_{t+1}), s_t(u)e^{-1/q}(1 - O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \leq \tilde{s}_t(u) \leq s_t(u)e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}))\}. \quad (8)$$

Lemma 14. *Given Assumption 1, we have*

$$Pr(\tilde{\mathcal{S}}) \geq 1 - e^{-\psi} O(n).$$

Proof. Suppose u is an uncolored vertex at the beginning of round t, and c a color in its palette.

$$\begin{aligned} & Pr\{c \text{ is removed from } S_t(u) \text{ in phase II.1}\} \\ & \quad = 1 - Pr\{\text{no neighbor of } u \text{ is assigned } c\} \\ & \quad = 1 - \prod_{v \in D_t(u,c)} (1 - Pr\{v \text{ is assigned } c\}) \\ & \quad = 1 - \prod_{v \in D_{u,c}} (1 - \frac{1}{q^2}\frac{1}{d_t}) \\ & \quad \leq 1 - (1 - \frac{1}{q^2}\frac{1}{d_t})^{d_t(u,c)} \\ & \quad \leq 1 - (1 - \frac{1}{q^2}\frac{1}{d_t})^{q d_t(1+e_t)} \\ & \quad \leq 1 - e^{\log(1-\frac{1}{q^2}\frac{1}{d_t}) q d_t(1+e_t)} && \langle \log(1+x) = x + O(x^2) \rangle \\ & \quad \leq 1 - e^{(-\frac{1}{q^2}\frac{1}{d_t} + O(\frac{1}{d_t})^2) q d_t(1+e_t)} && \langle \text{Assumption 1} \rangle \\ & \quad \leq 1 - e^{-1/q}(1 + O(e_t + \frac{1}{d_t})) \end{aligned}$$

In phase II.2 of round $t+1$ we remove colors from the palette using an appropriate bernoulli variable, to get

$$Pr\{c \notin \tilde{S}_t(u)\} = 1 - e^{-1/q}(1 + O(e_t + \frac{1}{d_t})). \quad (9)$$

Using linearity of expectation

$$\forall u \in V(G_{t+1}), E[\tilde{s}_t(u)] = s_t(u)e^{-1/q}(1 + O(e_t + \frac{1}{d_t})).$$

For concentration of $\tilde{s}_t(u)$, suppose $s_t(u) = m$. Let $c_1, \ldots, c_m$ be the colors in $S_t(u)$. Then $\tilde{S}_t(u)$ may be considered a random variable determined by m trials $T_1, \ldots, T_m$ where T_i is the set of vertices in G_t that are assigned color c_i in round t. Observe that T_i affects $\tilde{S}_t(u)$ by at most 1 given $T_1, \ldots, T_{i-1}$. Now using Theorem A we get,

$$Pr\{|\tilde{s}_t(u) - E[\tilde{s}_t(u)]| \geq \sqrt{\psi s_t(u)}\} \leq e^{-\psi}O(1).$$

We end the proof using the union bound for probabilities. □

We now focus on the sets $D_t(u, c)$. The following two lemmas will help.

Lemma 15. *Let u be an uncolored vertex, and c a color in its palette at the beginning of round t. Then given Assumption 1, we have*

$$Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\} = \frac{1}{q^2}\frac{1}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})).$$

Proof.

$$\begin{aligned}
&Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\} \\
&\quad = Pr\{u \text{ is assigned } c\}Pr\{c \in \tilde{S}_t(u)\} \\
&\quad = \frac{1}{q^2}\frac{1}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})) && \langle \text{Equation (9)} \rangle
\end{aligned}$$

□

The following lemma is a consequence of the previous one.

Lemma 16. *Let u be an uncolored vertex at the beginning of round t. Then given Assumption 1, we have*

$$Pr\{u \text{ is colored}\} \geq \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})).$$

Proof. Consider the event

$$\{u \text{ is colored}\} = \bigcup_{c \in S_t(u)} \{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\}.$$

Since the events in the union on the right hand side of the equation above are independent,

$$Pr\{u \text{ is colored}\} = 1 - \prod_{c \in S_t(u)} (1 - Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\}).$$

Now using Lemma 15, we get

$$\begin{aligned}
&Pr\{u \text{ is colored}\} \\
&\quad \geq 1 - (1 - \frac{1}{q^2}\frac{1}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})))^{s_t(u)} \\
&\quad \geq 1 - (1 - \frac{1}{q^2}\frac{1}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})))^{\frac{q-1}{q}s_t(1-e_t)} && \langle \text{Assumption 1} \rangle \\
&\quad \geq 1 - exp(-\frac{q-1}{q^3}\frac{s_t}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t}))) \\
&\quad \geq 1 - (1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t}))) && \langle \text{since } q \geq 2 \rangle \\
&\quad = \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q}(1 + O(e_t + \frac{1}{d_t})).
\end{aligned}$$

□

We will need the following definitions.

Definition 5.

- *Let $\tilde{D}_t(u,c)$ be the set of uncolored vertices that have color c in their palettes and are uncolored in round t, just before the cleanup phase. That is,*

$$\tilde{D}_t(u,c) = D_t(u,c) \setminus (\{v | c \notin \tilde{S}_t(v)\} \cup \{v | v \textit{ is colored in round } t\}).$$

- *Let $\tilde{d}_t(u,c)$ be the size of $\tilde{D}_t(u,c)$.*
- $\bar{d}_t(u) := \sum_{c\in\tilde{S}_t(u)} \tilde{d}_t(u,c) = \sum_{c\in S_t(u)} 1_{\{c\in\tilde{S}_t(u)\}}\tilde{d}_t(u,c)$
- $\tilde{d}_t(u) := \frac{\bar{d}_t(u)}{\tilde{s}_t(u)}$

Now consider the event

$$\tilde{\mathcal{A}} := \{\forall u \in V(G_{t+1}), \tilde{d}_t(u) \leq d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\}. \quad (10)$$

Lemma 17. *Given Assumption 1, we have*

$$Pr(\tilde{\mathcal{A}}) \geq 1 - e^{-\psi}O(n^2).$$

Proof. Let u be an uncolored vertex at the beginning of round t, and let c be a color in its palette. For a vertex v in $D_t(u,c)$, Lemma 15 implies that $Pr\{v \text{ is colored with } d\} = O(1/d_t)$ for any color d in $S_t(v)$. Thus,

$$\begin{aligned}
Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is colored}\}) &= \sum_{d\in S_t(v)} Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is colored with } d\}) \\
&= \sum_{d\in S_t(v)} Pr\{c \notin \tilde{S}_t(v) | v \text{ is colored with } d\} Pr\{v \text{ is colored with } d\} \\
&= Pr\{c \notin \tilde{S}_t(v)\}(1 + O(\frac{1}{d_t})) \sum_{d\in S_t(v)} Pr\{v \text{ is colored with } d\} \\
&= Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is colored}\}(1 + O(\frac{1}{d_t})).
\end{aligned}$$

A straightforward computation now shows that

$$Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is not colored}\}) = Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is not colored}\}(1 + O(\frac{1}{d_t})). \quad (11)$$

Now, v is removed from the set $D_t(u,c)$ if either it is colored or color c is removed from its palette. This means that event

$$\{v \notin \tilde{D}_t(u,c)\} = \{v \text{ is colored}\} \cup (\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is not colored}\}).$$

Since G is triangle-free, u and v do not have any common neighbors. This implies that

$$\begin{aligned}
&Pr\{v \notin \tilde{D}_t(u,c) | c \in \tilde{S}_t(u)\} \\
&\quad = Pr\{v \notin \tilde{D}_t(u,c)\}(1 + O(\frac{1}{d_t})) \\
&\quad = (Pr\{v \text{ is colored}\} + Pr(\{c \notin \tilde{S}_t(v)\} \cap \{v \text{ is not colored}\}))(1 + O(\frac{1}{d_t})) \\
&\quad = (Pr\{v \text{ is colored}\} + Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is not colored}\})(1 + O(\frac{1}{d_t})) && \langle\text{equation (11)}\rangle \\
&\quad = (Pr\{v \text{ is colored}\} + (1 - e^{-1/q})(1 - Pr\{v \text{ is colored}\}))(1 + O(e_t + \frac{1}{d_t})) && \langle\text{equation (9)}\rangle \\
&\quad = (1 - (1 - Pr\{v \text{ is colored}\})e^{-1/q})(1 + O(e_t + \frac{1}{d_t})) \\
&\quad \geq (1 - (1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q})(1 + O(e_t + \frac{1}{d_t})) && \langle\text{Lemma 16}\rangle.
\end{aligned}$$

Using linearity of expectation

$$E[\tilde{d}_t(u,c) | c \in \tilde{S}_t(u)] = E[\tilde{d}_t(u,c)](1 + O(\frac{1}{d_t})) \leq d_t(u,c)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \frac{1}{d_t})). \tag{12}$$

Now using the above bound

$$\begin{aligned}
E[\bar{d}_t(u)] &= \sum_{c \in S_t(u)} Pr\{c \in \tilde{S}_t(u)\} E[\tilde{d}_t(u,c) | c \in \tilde{S}_t(u)] \\
&\leq e^{-1/q} \sum_{c \in S_t(u)} d_t(u,c)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \frac{1}{d_t})) \\
&\leq e^{-1/q} s_t(u) d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \frac{1}{d_t}))
\end{aligned}$$

For concentration of $\bar{d}_t(u)$, suppose $s_t(u) = m$. Let $c_1, \ldots, c_m$ be the colors in $S_t(u)$. Then $\bar{d}_t(u)$ may be considered a random variable determined by the random trials $T_1, \ldots, T_m$, where T_i is the set of vertices in G_t that are assigned color c_i in round t. Observe that T_i affects $\bar{d}_t(u)$ by at most $d_t(u,c)$.

Thus $\sum \alpha_i^2$ in the statement of Theorem A is less that $\sum_{c \in S_t(u)} d_t^2(u,c)$. This upperbound is maximized when the $d_t(u,c)$ take the extreme values of qd_t and 0 subject to $d_t(u) = \frac{1}{s_t(u)} \sum_{c \in S_t(u)} d_t(u,c)$. Thus

$$\sum \alpha_i^2 \leq O((d_t)^2 d_t(u) s_t(u)/d_t) \leq O(s_t(u) d_t d_t(u))$$

Using Theorem A, we get

$$\begin{aligned}
&Pr\{\bar{d}_t(u) - e^{-1/q} s_t(u) d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \frac{1}{d_t})) \geq O(\sqrt{\psi s_t(u) d_t d_t(u)})\} \\
&\qquad \leq e^{-\psi} O(1).
\end{aligned}$$

Lemma 14 says that the event $\tilde{\mathcal{S}}$ occurs with probability $1 - e^{-\psi}O(n)$. Thus

$$\begin{aligned}
&Pr\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)} - d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \frac{1}{d_t} + \sqrt{\frac{\psi}{s_t}})) \geq O(\sqrt{\frac{\psi d_t d_t(u)}{s_t}})\} \\
&\qquad \leq e^{-\psi} O(n).
\end{aligned}$$

Therefore

$$Pr\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)} \geq d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\} \leq e^{-\psi}O(n).$$

We end the proof using the union bound for probabilities. □

Note that $\frac{\bar{d}_t(u)}{\tilde{s}_t(u)}$ is the average $|\tilde{D}_t(u,c)|$ at a vertex u at the end phase II. Phase III only brings this average down by removing colors with large $d_{u,c}$. Thus we get the next lemma almost immediately. Consider the event

$$\mathcal{A} := \{\forall u \in V(G_{t+1}), d_{t+1}(u) \leq d_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})\}.$$

Lemma 18. *Given Assumption 1, we have*

$$Pr(\mathcal{A}) \geq 1 - e^{-\psi}O(n^2).$$

Proof. Assume the occurrence event $\tilde{\mathcal{A}}$, as defined in equation (10), and let u be a vertex in $V(G_{t+1})$. Then

$$\begin{aligned}
&d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\\
&\quad = d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) + O(\sqrt{\frac{\psi d_t(u) d_t}{s_t}})\\
&\quad = d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) + d_t O(\sqrt{\frac{\psi d_t(u)}{s_t d_t}})\\
&\quad \leq d_t(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) + d_t O(\sqrt{\frac{\psi}{s_t}})\\
&\quad = d_t(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})).
\end{aligned}$$

Thus the event

$$\begin{aligned}
&\{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)} \geq d_t(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}))\}\\
&\quad \subseteq \{\frac{\bar{d}_t(u)}{\tilde{s}_t(u)} \geq d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\}.
\end{aligned}$$

Therefore

$$Pr(\mathcal{A}) \geq Pr(\tilde{\mathcal{A}}) \geq e^{-\psi}O(n^2).$$

□

Next we show that in the cleanup phase of round t, no vertex discards so many colors that its palette size in round $t+1$ becomes less than $\frac{q-1}{q}s_{t+1}(1 - e_{t+1})$. Consider the event

$$\mathcal{S} := \{\forall v \in V(G_{t+1}), \exists \alpha \in [0, \frac{1}{q}] \text{ such that} \tag{13}$$

$$s_{t+1}(u) \geq (1-\alpha)s_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})), \tag{14}$$

$$d_{t+1}(u) \leq \frac{1 - q\alpha}{1 - \alpha}d_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}))\}. \tag{15}$$

Lemma 19. *Given Assumption 1, we have*

$$Pr(\mathcal{S}) \geq 1 - e^{-\psi}O(n^2).$$

Proof. Consider vertex $u \in V(G_{t+1})$. Using Assumption 1, at round t, $\exists \alpha \in [0, \frac{1}{q}]$ such that $s_t(u) \geq (1-\alpha)s_t(1-e_t)$ and $d_t(u) \leq \frac{1-q\alpha}{1-\alpha}d_t(1+e_t)$. Assuming the occurrence of event $\tilde{\mathcal{S}}$, as defined in equation (8), we get

$$\begin{aligned}
\tilde{s}_t(u) &= s_t(u)e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \\
&\geq (1-\alpha)s_t e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \\
&\geq (1-\alpha)s_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})).
\end{aligned}$$

Assuming event $\tilde{\mathcal{A}}$ occurs,

$$\begin{aligned}
\tilde{d}_t(u) &\leq d_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \\
&\leq \frac{1-q\alpha}{1-\alpha}d_t(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})e^{-1/q}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \\
&\leq \frac{1-q\alpha}{1-\alpha}\gamma d_{t+1}.
\end{aligned}$$

where γ is the smallest number in $[1, \infty)$ for which the above inequality is true. Combining the preceding inequalities, we get

$$\gamma = 1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}).$$

In the cleanup phase of our algorithm(given in Section 2.2), the change in palette is equivalent to the following process.

1. Add $\frac{\alpha}{1-\alpha}\tilde{s}_t(u)$ arbitrary colors to u's palette, with $d_{u,c} = q\gamma d_{t+1}$. This adjusts the palette size to $\tilde{s}_t(u) \geq s_{t+1}(1 + O(e_t + \sqrt{\psi/s_t} + 1/d_t))$. Lemma 12 ensures that the adjusted new average is $\tilde{d}_t(u) \leq \gamma d_{t+1}$
2. Remove all the colors with $d_t(u,c) \geq q\gamma d_{t+1}$.

Now we use Lemma 11, setting μ to γd_{t+1} and $q\mu$ to $q\gamma d_{t+1}$, to get

$$s_{t+1}(u) \geq (1-\alpha)s_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}))$$

and

$$d_{t+1}(u) \leq \frac{1-q\alpha}{1-\alpha}d_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})).$$

The result is obtained using Lemmas 14 and 17 to get

$$Pr(\tilde{\mathcal{S}} \cap \tilde{\mathcal{A}}) \geq 1 - e^{-\psi}O(n^2),$$

and noting that the preceding inequalities for $s_{t+1}(u)$ and $d_{t+1}(u)$ are true for every vertex u in $V(G_{t+1})$, given events $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{A}}$ occur.

□

Now consider the event

$$\mathcal{D} := \{\forall v \in V(G_{t+1}), \eta_{t+1}(v) \leq \eta_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{d_t}} + \frac{1}{d_t}))\}.$$

Lemma 20. *Given Assumption 1, we have*

$$Pr(\mathcal{D}) \geq 1 - e^{-\psi}O(n).$$

Proof. Suppose u is an uncolored vertex at the beginning of round t. By Lemma 16 we have

$$Pr\{u \text{ is colored}\} \geq \tfrac{q-1}{2q^3}\tfrac{s_t}{d_t}e^{-1/q}(1 + O(e_t + \tfrac{1}{d_t})).$$

Using linearity of expectation, $\forall u$ in $V(G_{t+1})$

$$\begin{aligned} E[\eta_{t+1}(u)] &\leq \eta_t(u)(1 - \frac{q-1}{2q^3}\frac{s_t}{d_t}e^{-1/q})(1 + O(e_t + \frac{1}{d_t})) && \langle\text{Lemma 13}\rangle \\ &\leq \eta_{t+1}(1 + O(e_t + \frac{1}{d_t})). \end{aligned}$$

We again resort to Theorem A to study concentration of $\eta_{t+1}(u)$. Suppose $\eta_t(u) = m$. Let $v_1, \ldots, v_m$ be the vertices in $N_t(u)$. Then $\eta_{t+1}(u)$ may be considered a random variable determined by $T_1, \ldots, T_m$, where T_i is a random variable which indicates that v_i is colored in round t or not. The affect of each T_i given $T_1, \ldots, T_{i-1}$ is at most $O(\frac{s_t}{d_t})$. Thus $\sum \alpha_i^2$ in the statement of Theorem A is

$$O(\frac{\eta_t s_t^2}{d_t^2}) = O(\frac{\eta_t s_t}{d_t}).$$

Using Theorem A, we get

$$Pr\{\eta_{t+1}(u) \geq \eta_{t+1}(1 + O(e_t + \frac{1}{d_t})) + O(\sqrt{\psi\frac{s_t\eta_t}{d_t}})\} \leq e^{-\psi}.$$

Thus

$$Pr\{\eta_{t+1}(u) \geq \eta_{t+1}(1 + O(e_t + \sqrt{\psi\frac{s_t}{d_t\eta_t}} + \frac{1}{d_t}))\} \leq e^{-\psi}.$$

Since the above is true for every vertex u in $V(G_{t+1})$, the theorem is proved by applying the union bound on probabilties. ☐

5.2 Expected Values and Concentration Inequalities for the Second Stage

We now focus on the second stage of the algorithm, where $s_t \geq q^2 d_t$. The pattern of analysis for the second stage is similar to that of the first stage. But simply reproducing the previous section, with changes here and there, would make it difficult to understand the differences. With this in mind, we proceed much faster now and focus on the expectations of variables, leaving out all concentration calculations. These can be filled in by using the corresponding proofs in Section 5.1 as templates.

Consider now the event

$$\tilde{\mathcal{S}} := \{\forall u \in V(G_{t+1}), s_t(u)e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 - O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t})) \leq \tilde{s}_t(u) \leq s_t(u)e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t}))\}.$$

Lemma 21. *Given Assumption 1, we have*

$$Pr(\tilde{\mathcal{S}}) \geq 1 - e^{-\psi}O(n).$$

Proof. Suppose u is an uncolored vertex at the beginning of round t, and c a color in its palette.

$$\begin{aligned} Pr\{c \text{ is removed from } S_t(u) \text{ in phase II.1}\} &= 1 - \prod_{v \in D_t(u,c)} (1 - Pr\{v \text{ is assigned } c\}) \\ &= 1 - \prod_{v \in D_{u,c}} (1 - \frac{1}{q^2}\frac{1}{s_t}(1 + O(e_t + \frac{1}{s_t}))) \\ &\leq 1 - e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})) \end{aligned}$$

In phase II.2 we remove colors from the palette using an appropriate bernoulli variable, to get

$$Pr\{c \notin \tilde{S}_t(u)\} = 1 - e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})).$$

Using linearity of expectation

$$\forall u \in V(G_{t+1}), E[\tilde{s}_t(u)] = s_t(u)e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})).$$

The rest of the proof follows that of Lemma 14. □

We now focus on the sets $D_t(u, c)$. The following two lemmas will help.

Lemma 22. *Let u be an uncolored vertex at the beginning of round t, and c a color in its palette. Then given Assumption 1, we have*

$$Pr\{u \textit{ is assigned } c \textit{ and } c \in \tilde{S}_t(u)\} \geq \frac{1}{q^2}\frac{1}{s_t}e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})).$$

Proof.

$$\begin{aligned} Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\} &= Pr\{u \text{ is assigned } c\}Pr\{c \in \tilde{S}_t(u)\} \\ &= \frac{1}{q^2}\frac{1}{s_t}\prod_{v \in D_t(u,c)}(1 - \frac{1}{q^2}\frac{1}{s_t}) \\ &\geq \frac{1}{q^2}\frac{1}{s_t}e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})) \end{aligned}$$

□

The following lemma is a consequence of the previous one.

Lemma 23. *Let u be an uncolored vertex at the beginning of round t. Then given Assumption 1, we have*

$$Pr\{u \textit{ is colored}\} \geq \frac{q-1}{2q^3}e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})).$$

Proof. Following the proof of Lemma 16, but using Lemma 22 instead of Lemma 15, we get

$$\begin{aligned} Pr\{u \text{ is colored}\} &\geq 1 - (1 - \frac{1}{q^2}\frac{1}{s_t}e^{\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})))^{s_t(u)} \\ &\geq \frac{q-1}{2q^3}e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})). \end{aligned}$$

□

Consider the event

$$\tilde{\mathcal{A}} := \{\forall u \in V(G_{t+1}), \tilde{d}_t(u) \leq d_t(u)(1 - \frac{q-1}{2q^3}e^{-\frac{1}{q}\frac{d_t}{s_t}})e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}}))\}.$$

Lemma 24. *Given Assumption 1, we have*

$$Pr(\tilde{\mathcal{A}}) \geq 1 - e^{-\psi}O(n^2).$$

Proof. Let u be an uncolored vertex at the beginning of round t, and let c be a color in its palette. For a vertex v in $D_t(u, c)$, the event

$$\{v \notin \tilde{D}_t(u, c)\} = \{v \text{ is colored}\} \cup (\{v \text{ is not colored}\} \cap \{c \notin \tilde{S}_t(v)\}).$$

Then as in the proof of Lemma 17, we get

$$\begin{aligned} Pr\{v \notin \tilde{D}_t(u,c) | c \in \tilde{S}_t(u)\} &= Pr\{v \notin \tilde{D}_t(u,c)\}(1 + O(\frac{1}{s_t})) \\ &= (Pr\{v \text{ is colored}\} + (1 - e^{-\frac{1}{q}\frac{d_t}{s_t}})(1 - Pr\{v \text{ is colored}\}))(1 + O(\frac{1}{s_t})) \\ &= (1 - (1 - Pr\{v \text{ is colored}\})e^{-\frac{1}{q}\frac{d_t}{s_t}})(1 + O(e_t + \frac{1}{s_t})) \\ &\geq (1 - (1 - \frac{q-1}{2q^3}e^{-\frac{1}{q}\frac{d_t}{s_t}})e^{-\frac{1}{q}\frac{d_t}{s_t}})(1 + O(e_t + \frac{1}{s_t})). \end{aligned}$$

By linearity of expectation, $\forall u \in V(G_{t+1}), \forall c \in \tilde{s}_t(u)$

$$E[\tilde{d}_t(u,c) | c \in \tilde{s}_t(u)] \leq E[\tilde{d}_t(u,c)](1 + O(\frac{1}{s_t})) \leq d_t(u,c)(1 - \frac{q-1}{2q^3}e^{-\frac{1}{q}\frac{d_t}{s_t}})e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})).$$

Now using the above bound

$$\begin{aligned} E[\bar{d}_t(u)] &= \sum_{c \in s_t(u)} Pr\{c \in \tilde{s}_t(u)\} E[\tilde{d}_t(u,c) | c \in \tilde{s}_t(u)] \\ &\leq e^{-\frac{1}{q}\frac{d_t}{s_t}} \sum_{c \in s_t(u)} d_t(u,c)(1 - \frac{q-1}{2q^3}e^{-\frac{1}{q}\frac{d_t}{s_t}})e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})) \\ &\leq e^{-\frac{1}{q}\frac{d_t}{s_t}} s_t(u) d_t(u)(1 - \frac{q-1}{2q^3}e^{-\frac{1}{q}\frac{d_t}{s_t}})e^{-\frac{1}{q}\frac{d_t}{s_t}}(1 + O(e_t + \frac{1}{s_t})). \end{aligned}$$

The rest of the proof follows that of Lemma 17. □

Consider the event

$$\mathcal{A} := \{\forall u \in V(G_{t+1}) d_{t+1}(u) \leq d_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t})\}.$$

Lemma 25. *Given Assumption 1, we have*

$$Pr(\mathcal{A}) \geq 1 - e^{-\psi}O(n^2).$$

Proof. The proof follows that of Lemma 18. □

Consider the event

$$\begin{aligned} \mathcal{S} := \{\forall v \in V(G_{t+1}), &\exists \alpha \in [0, \frac{1}{q}] \text{ such that} \\ &s_{t+1}(u) \geq (1 - \alpha)s_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t})), \\ &d_{t+1}(u) \leq \frac{1 - q\alpha}{1 - \alpha} d_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{s_t}} + \frac{1}{s_t}))\}. \end{aligned}$$

Lemma 26. *Given Assumption 1, we have*

$$Pr(\mathcal{S}) \geq 1 - e^{-\psi} O(n^2).$$

Proof. The proof follows that of Lemma 19. $\square$

Now consider the event

$$\mathcal{D} := \{\forall v \in V(G_{t+1}), \eta_{t+1}(v) \leq \eta_{t+1}(1 + O(e_t + \sqrt{\frac{\psi}{d_t}}))\}.$$

Lemma 27. *Given Assumption 1, we have*

$$Pr(\mathcal{D}) \geq 1 - e^{-\psi} O(n)$$

Proof. Using Lemma 23

$$Pr\{u \text{ is colored}\} \geq \tfrac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}} (1 + O(e_t)).$$

Using linearity of expectation, $\forall u$ in G_{t+1}

$$\begin{aligned} E[\eta_{t+1}(u)] &\leq \eta_t(u)(1 - \frac{q-1}{2q^3} e^{-\frac{1}{q}\frac{d_t}{s_t}} (1 + O(e_t)) \\ &\leq \eta_{t+1}(1 + O(e_t)). \end{aligned}$$

The rest of the proof follows that of Lemma 20. $\square$

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